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State Estimation With Finite Signal-to-Noise Models via Linear Matrix Inequalities

This paper presents estimation design methods for linear systems whose white noise sources have intensities affinely related to the variance of the signal they corrupt. Systems with such noise sources have been called finite signal-to-noise (FSN) models, and the results provided in prior work demonstrate that estimation problem for FSN systems (estimating to within a specified covariance error bound) is nonconvex. We shall show that a mild additional constraint for scaling will make the problem convex. In this paper, sufficient conditions for the existence of the state estimator are provided; these conditions are expressed in terms of linear matrix inequalities (LMIs), and the parametrization of all admissible solutions is provided. Finally, a LMI-based estimator design is formulated, and the performance of the estimator is examined by means of numerical examples.
[DOI: 10.1115/1.2432358]

1 Introduction

The traditional noise model used in estimation and control theory is white noise, whose intensity is independent of the variance of the signal it is corrupting. However, in many engineering applications, this traditional noise model has serious deficiencies. As an example, the level of turbulence on an aircraft increases with the angle of attack, where surface error leads to turbulence generators. A new noise model, the finite signal-to-noise (FSN) model, which fits many practical situations, was proposed in [1–3], where the intensity of the noise corrupting the signal depends affinely on the variance of that signal.

FSN noise models are more practical than normal white noise models because they allow the variances of the noises to be affinely related to the variances of the signals they corrupt. Such FSN noises are found in digital signal processing with both fixed-point and floating-point arithmetic. Yet it is known that the influence of round-off errors are realization dependent [4], larger signals suffer larger computational errors; hence, increasing the number of bits in the computation does not solve the finite precision computing problem. Such FSN models are found in analog sensors and actuators [5], which produce more noise when the power supplies in these devices must provide more power (for an increased dynamic range of the signals in the estimation or control problem). Therefore, both the research to solve the finite precision computing problem and the research to create novel system design theories would benefit from the FSN models.

One important benefit of the FSN model in a linear control problem is that it keeps the control finite at the maximal accuracy, since a larger control signal comes at the price of larger noise, which degrades the system performance. This is in contrast to LQG theory, where the maximal accuracy is achieved at infinite control gain, which has a greater tendency to destabilize the system dynamics. Therefore, the FSN model has a significant effect on the robustness of the controller [3,6,7].

Recent studies also show the use of the FSN model for economic system design [8]. Today, most control system design begins with a selection of components, and many engineering problems involve economic considerations, especially in mechanical and biochemical engineering. Since the signal-to-noise ratio is one way to define precision, assuming that the component cost is proportional to its signal-to-noise ratio, it is reasonable to integrate the instrumentation and control design, to obtain a low-cost system for given performance requirements.

Furthermore, the FSN noise model has also been widely used in studying biological movements. The substantial variability of biological movements indicates that the sensory-motor system operates in the presence of large disturbances. Noise in the motor commands described the uncertainty of actual force produced by the muscle, which will lead to movement inaccuracy and variability. By studying the goal-directed eye and arm movements, Harris and Wolpert, in 1998, proposed that the noise in the motor commands is also signal dependent [9]. This is a very important assumption, which is consistent with the observation captured by the empirical Fitts' law. Control of such systems should obviously take this phenomenon into account because an appropriately chosen control signal can actually decrease the noise.

There has been great effort in recent years to provide a control theory for the FSN model. See [1,6,7] for a discussion of control problems with FSN noise models. Since the FSN model reflects more realistic properties in engineering, as well as neuroscience [9–13], a complete theory, which includes control and estimation, should be developed.

This paper focuses on the study of estimation problem for the FSN model. Reference [3] demonstrates that the estimation problem is nonconvex. We shall show that a mild additional constraint for scaling will make the problem convex. The basic problem solved is to find a state estimator that bounds the estimation error below a specified error covariance.

The paper is organized as follows. In Sec. 2, the estimation problem for the FSN model is formulated. In Sec. 3, sufficient conditions for the existence of the state estimator are given. Section 4 derives a linear estimator subject to a performance requirement; a numerical example is presented, and the comparison between the FSN filter design and the Kalman filter is discussed. The discrete-time counterparts of the continuous-time results presented in Secs. 3 and 4 are given in Sec. 5, where two more numerical examples are presented. Some concluding remarks are drawn in Sec. 6.

2 System Model and Problem Formulation

Consider the following linear system with state space representation

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Contributed by the Dynamic Systems, Measurement, and Control Division of ASME for publication in the JOURNAL OF DYNAMIC SYSTEMS, MEASUREMENT, AND CONTROL. Manuscript received November 28, 2005; final manuscript received July 15, 2006. Assoc. Editor: Jordan Berg. Paper presented at the 2003, The 42nd IEEE Conference on Decision and Control.

$$\dot{x} = Ax + D\omega \quad (1)$$

$$y = Cx + v \quad (2)$$

$$z = C_z x \quad (3)$$

where $x \in \mathbf{R}^{n_x}$ is state variable, $y \in \mathbf{R}^{n_y}$ is the measurement output, and $z \in \mathbf{R}^{n_z}$ is the output of interest for performance evaluation; $\omega \in \mathbf{R}^{n_w}$ and $v \in \mathbf{R}^{n_y}$ are zero-mean FSN white noises with unknown intensities W, V, respectively; A, C, C_z, and D are given constant matrices that have proper dimensions. Here we consider that the noise source is modeled according to the FSN assumption, where the intensity of the noise corrupting a signal is proportional to the intensity of that signal. That is, assuming

$$\varepsilon_\infty\{\omega(t)\} = 0, \quad \varepsilon_\infty\{\omega(t)\omega(\tau)^T\} = W\delta(t-\tau) \quad (4)$$

$$\varepsilon_\infty\{v(t)\} = 0, \quad \varepsilon_\infty\{v(t)v(\tau)^T\} = V\delta(t-\tau) \quad (5)$$

where for a given stationary stochastic process $\omega(t)$, the notation $\varepsilon_\infty\{\omega(t)\omega(t)^T\}$ denotes the asymptotic operation $\lim_{t\to\infty}\varepsilon\{\omega(t)\omega(t)^T\}$.

Suppose the vector $\omega_a \in \mathbf{R}^{n_w}$ describes the signal that is corrupted by the noise ω, and ω_a is linearly related to the state variable x

$$\omega_a = Mx \quad (6)$$

where matrix M is given. By defining the state covariance matrix

$$X = \varepsilon_\infty\{x(t)x(t)^T\} \quad (7)$$

we obtain the intensities of noise ω and v as

$$W = W_0 + \Sigma_\omega M X M^T \Sigma_\omega \quad (8)$$

$$V = V_0 + \Sigma_v C X C^T \Sigma_v \quad (9)$$

where W_0, V_0 are given positive definite constant matrices, and

$$\Sigma_\omega = \mathrm{diag}\{\sigma_{\omega_1}, \sigma_{\omega_2}, \ldots, \sigma_{\omega_{n_\omega}}\} \quad (10)$$

$$\Sigma_v = \mathrm{diag}\{\sigma_{v_1}, \sigma_{v_2}, \ldots, \sigma_{v_{n_y}}\} \quad (11)$$

where $\sigma_{\omega_i}, \sigma_{v_i}$ are noise-to-signal ratio (NSR) of the ith channel, respectively.

For this system, the objective is to design a linear filter with state space representation

$$\dot{\hat{x}} = A\hat{x} + F(y - C\hat{x}) \quad (12)$$

$$\hat{z} = C_z \hat{x} \quad (13)$$

where $\hat{x}$ is the estimate of the state x, F is the filter gain to be determined such that $(A-FC)$ is asymptotically stable, and the estimation error has covariance less than a specified matrix. The estimation error is $\tilde{x}=x-\hat{x}$, and the estimation error system is given by

$$\dot{\tilde{x}} = (A - FC)\tilde{x} + D\omega - Fv \quad (14)$$

$$\tilde{z} = C_z \tilde{x} \quad (15)$$

where $\tilde{z}$ denotes the estimation error of particular interests. The key idea of this filtering problem is to find the estimate $\hat{x}$ of x such that the performance criterion $\varepsilon_\infty\{\tilde{z}\tilde{z}^T\} < \Omega$ is satisfied.

In this paper, the following two problems are analyzed. First, we will explore the existence condition of the state estimator. We will be able to provide the sufficient conditions for the existence of the state estimator based on linear matrix inequalities (LMIs). Second, we will determine if there exists a filter gain F such that $\varepsilon_\infty\{\tilde{z}\tilde{z}^T\} < \Omega$ is satisfied for the given Ω.

3 Existence Condition

In the following, we consider the augmented estimation error dynamics

$$\dot{\mathbf{x}} = \mathcal{A}\mathbf{x} + \mathcal{D}\mathbf{w}, \quad (16)$$

where

$$\mathbf{x} = \begin{pmatrix} \tilde{x} \\ \hat{x} \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} \omega \\ v \end{pmatrix} \quad (17)$$

$$\mathcal{A} = \begin{pmatrix} A - FC & 0 \\ FC & A \end{pmatrix} = \mathcal{A}_0 + \mathcal{B}_0 F \mathcal{C}_0 \quad (18)$$

$$\mathcal{D} = \begin{pmatrix} D & -F \\ 0 & F \end{pmatrix} = \mathcal{D}_0 + \mathcal{B}_0 F E_0 \quad (19)$$

$$\mathcal{A}_0 = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, \quad \mathcal{B}_0 = \begin{pmatrix} -I \\ I \end{pmatrix} \quad (20)$$

$$\mathcal{C}_0 = (C \quad 0), \quad \mathcal{D}_0 = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}, \quad E_0 = (0 \quad I) \quad (21)$$

We start by defining the upper bound of the state covariance matrix of system (16) as

$$\mathcal{X} \geq \varepsilon_\infty\{\mathbf{x}(t)\mathbf{x}(t)^T\} \quad (22)$$

if it exists, it satisfies the following inequality:

$$0 > \mathcal{X}\mathcal{A}^T + \mathcal{A}\mathcal{X} + \mathcal{D}\begin{pmatrix} W & 0 \\ 0 & V \end{pmatrix}\mathcal{D}^T \quad (23)$$

Definition 1 (*Mean Square Stable*). *The error system* (16) *with FSN noise inputs is mean square stable if there exists a positive definite covariance matrix* $\mathcal{X}$ *satisfying the inequality* (23).

Substitution of (8), (9), (18), (19) into the above inequality, yields

$$0 > \mathcal{X}(\mathcal{A}_0 + \mathcal{B}_0 F \mathcal{C}_0)^T + (\mathcal{A}_0 + \mathcal{B}_0 F \mathcal{C}_0)\mathcal{X} + \mathcal{N}\mathcal{X}\mathcal{N}^T + (\mathcal{B}_0 F G_0)\mathcal{X}(\mathcal{B}_0 F G_0)^T + (\mathcal{D}_0 + \mathcal{B}_0 F E_0)\mathcal{W}(\mathcal{D}_0 + \mathcal{B}_0 F E_0)^T \quad (24)$$

where

$$\mathcal{N} = \begin{pmatrix} D\Sigma_\omega M & D\Sigma_\omega M \\ 0 & 0 \end{pmatrix} \quad (25)$$

$$\mathcal{W} = \begin{pmatrix} W_0 & 0 \\ 0 & V_0 \end{pmatrix} \quad (26)$$

$$G_0 = (-\Sigma_v C \quad -\Sigma_v C) \quad (27)$$

LEMMA 1. (*assume* $\mathcal{W}=I$) *The inequality of* (24) *can be written as*

$$\Gamma F \Lambda + (\Gamma F \Lambda)^T + \Theta < 0 \quad (28)$$

where

$$\Theta = \begin{pmatrix} \mathcal{A}_0\mathcal{X} + \mathcal{X}\mathcal{A}_0^T + \mathcal{N}\mathcal{X}\mathcal{N}^T & 0 & \mathcal{D}_0 \\ 0 & -\mathcal{X} & 0 \\ \mathcal{D}_0^T & 0 & -I \end{pmatrix} \quad (29)$$

$$\Gamma = \begin{pmatrix} \mathcal{B}_0 \\ 0 \\ 0 \end{pmatrix} \quad (30)$$

$$\Lambda = (\mathcal{C}_0\mathcal{X} \quad G_0\mathcal{X} \quad E_0) \quad (31)$$

Proof. By using the Schur complement formula, the inequality (24) can be written as

$$\begin{pmatrix} \mathcal{X}\mathcal{A}^T + \mathcal{A}\mathcal{X} + \mathcal{N}\mathcal{X}\mathcal{N}^T & (\mathcal{B}_0 F G_0)\mathcal{X} & \mathcal{D}_0 + \mathcal{B}_0 F E_0 \\ \mathcal{X}(\mathcal{B}_0 F G_0)^T & -\mathcal{X} & 0 \\ (\mathcal{D}_0 + \mathcal{B}_0 F E_0)^T & 0 & -I \end{pmatrix} < 0$$

where $\mathcal{A}$ is defined in (18). Breaking the above matrix into two matrices and substituting (29) into the above inequality yields

$$\Theta + \begin{pmatrix} (\mathcal{B}_0 F \mathcal{C}_0)\mathcal{X} + \mathcal{X}(\mathcal{B}_0 F \mathcal{C}_0)^T & (\mathcal{B}_0 F G_0)\mathcal{X} & \mathcal{B}_0 F E_0 \\ \mathcal{X}(\mathcal{B}_0 F G_0)^T & 0 & 0 \\ (\mathcal{B}_0 F E_0)^T & 0 & 0 \end{pmatrix} < 0$$

Rewrite the above inequality as follows:

$$\Theta + \begin{pmatrix} \mathcal{B}_0 \\ 0 \\ 0 \end{pmatrix} F(\mathcal{C}_0\mathcal{X} \quad G_0\mathcal{X} \quad E_0) + \left[\begin{pmatrix} \mathcal{B}_0 \\ 0 \\ 0 \end{pmatrix} F(\mathcal{C}_0\mathcal{X} \quad G_0\mathcal{X} \quad E_0)\right]^T < 0$$

With the definition of Γ and Λ given in (30) and (31), the above condition can be equivalently written as (28). ■

In order to find the existence conditions of the state estimator and the parametrization of all the solutions, the following lemma from [14] can be applied.

LEMMA 2 (FINSLER'S LEMMA). *Let $x \in \mathbf{R}^n$, $\Theta = \Theta^T \in \mathbf{R}^{n\times n}$, $\Gamma \in \mathbf{R}^{n\times m}$, and $\Lambda \in \mathbf{R}^{k\times n}$. For a given matrix Γ with rank r, let $\Gamma^\perp \in \mathbf{R}^{(n-r)\times n}$ be an orthogonal complement of Γ such that $\Gamma^\perp\Gamma = 0$ and $\Gamma^\perp\Gamma^{\perp T} > 0$. Let $\Lambda^{T\perp}$ be any appropriate matrix such that $\Lambda^{T\perp}\Lambda^T = 0$ and $\Lambda^{T\perp}\Lambda^{T\perp T} > 0$. The following statements are equivalent*:

$$i.\ x^T\Theta x < 0, \quad \forall\, \Gamma^T x = 0, \quad \Lambda x = 0, \quad x \neq 0 \tag{32}$$

$$ii.\ \Gamma^\perp\Theta\Gamma^{\perp T} < 0 \tag{33}$$

$$\Lambda^{T\perp}\Theta\Lambda^{T\perp T} < 0 \tag{34}$$

$$iii.\ \exists\, \mu_1, \mu_2 \in \mathbf{R}: \Theta - \mu_1\Gamma\Gamma^T < 0 \tag{35}$$

$$\Theta - \mu_2\Lambda^T\Lambda < 0 \tag{36}$$

$$iv.\ \exists\, F \in \mathbf{R}^{m\times k}: \Gamma F\Lambda + (\Gamma F\Lambda)^T + \Theta < 0 \tag{37}$$

Finsler's lemma is a specialized version of the projection lemma [14], and it can be applied to obtain LMI formulations in control and estimation theory. By applying the Finsler's lemma, we obtain the following theorem.

LEMMA 3. *Condition* (28) *is equivalent to the following statement*: *there exist symmetric positive definite matrices $\mathcal{X}, P \in \mathbf{R}^{2n_x\times 2n_x}$ that satisfy*

$$\mathcal{X}P = I \tag{38}$$

$$\mathcal{B}_0^\perp(\mathcal{A}_0\mathcal{X} + \mathcal{X}\mathcal{A}_0^T + \mathcal{N}\mathcal{X}\mathcal{N}^T + \mathcal{D}_0\mathcal{D}_0^T)\mathcal{B}_0^{\perp T} < 0 \tag{39}$$

$$\begin{pmatrix} \mathcal{C}_0^T \\ G_0^T \\ E_0^T \end{pmatrix}^\perp \begin{pmatrix} P\mathcal{A}_0 + \mathcal{A}_0^T P + P\mathcal{N}\mathcal{X}\mathcal{N}^T P & 0 & P\mathcal{D}_0 \\ 0 & -P & 0 \\ \mathcal{D}_0^T P & 0 & -I \end{pmatrix} \begin{pmatrix} \mathcal{C}_0^T \\ G_0^T \\ E_0^T \end{pmatrix}^{\perp T} < 0 \tag{40}$$

Proof. The result follows from Lemma 1 and Finsler's lemma, where we note that

$$\Gamma^\perp = \begin{pmatrix} \mathcal{B}_0 \\ 0 \\ 0 \end{pmatrix}^\perp = \begin{pmatrix} \mathcal{B}_0^\perp & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}$$

Substituting the above equation and (29) into (33) yields

$$\begin{pmatrix} \mathcal{B}_0^\perp(\mathcal{A}_0\mathcal{X} + \mathcal{X}\mathcal{A}_0^T + \mathcal{N}\mathcal{X}\mathcal{N}^T)\mathcal{B}_0^{\perp T} & 0 & \mathcal{B}_0^\perp\mathcal{D}_0 \\ 0 & -\mathcal{X} & 0 \\ \mathcal{D}_0^T\mathcal{B}_0^{\perp T} & 0 & -I \end{pmatrix} < 0$$

A Schur complement of this matrix is

$$\begin{pmatrix} \mathcal{B}_0^\perp(\mathcal{A}_0\mathcal{X} + \mathcal{X}\mathcal{A}_0^T + \mathcal{N}\mathcal{X}\mathcal{N}^T)\mathcal{B}_0^{\perp T} & 0 \\ 0 & -\mathcal{X} \end{pmatrix} + \begin{pmatrix} \mathcal{B}_0^\perp\mathcal{D}_0 \\ 0 \end{pmatrix} (\mathcal{D}_0^T\mathcal{B}_0^{\perp T} \quad 0)$$
$$< 0$$

therefore

$$\begin{pmatrix} \mathcal{B}_0^\perp(\mathcal{A}_0\mathcal{X} + \mathcal{X}\mathcal{A}_0^T + \mathcal{N}\mathcal{X}\mathcal{N}^T + \mathcal{D}_0\mathcal{D}_0^T)\mathcal{B}_0^{\perp T} & 0 \\ 0 & -\mathcal{X} \end{pmatrix} < 0$$

which is equivalent to $\mathcal{B}_0^\perp(\mathcal{A}_0\mathcal{X} + \mathcal{X}\mathcal{A}_0^T + \mathcal{N}\mathcal{X}\mathcal{N}^T + \mathcal{D}_0\mathcal{D}_0^T)\mathcal{B}_0^{\perp T} < 0, \mathcal{X} > 0$.

Furthermore, since

$$\Lambda^{T\perp} = \begin{pmatrix} \mathcal{X}\mathcal{C}_0^T \\ \mathcal{X}G_0^T \\ E_0^T \end{pmatrix}^\perp = \begin{pmatrix} \mathcal{C}_0^T \\ G_0^T \\ E_0^T \end{pmatrix}^\perp \begin{pmatrix} \mathcal{X}^{-1} & 0 & 0 \\ 0 & \mathcal{X}^{-1} & 0 \\ 0 & 0 & I \end{pmatrix}$$

substituting (29) and the above equation into (34), yields

$$\begin{pmatrix} \mathcal{C}_0^T \\ G_0^T \\ E_0^T \end{pmatrix}^\perp \begin{pmatrix} \mathcal{X}^{-1}\mathcal{A}_0 + \mathcal{A}_0^T\mathcal{X}^{-1} + \mathcal{X}^{-1}\mathcal{N}\mathcal{X}\mathcal{N}^T\mathcal{X}^{-1} & 0 & \mathcal{X}^{-1}\mathcal{D}_0 \\ 0 & -\mathcal{X}^{-1} & 0 \\ \mathcal{D}_0^T\mathcal{X}^{-1} & 0 & -I \end{pmatrix}$$
$$\times \begin{pmatrix} \mathcal{C}_0^T \\ G_0^T \\ E_0^T \end{pmatrix}^{\perp T} < 0$$

By defining $\mathcal{X}^{-1} = P$, we could obtain (40) immediately.

The above theorem provides the existence condition for the state estimator, and the characterization given in Lemma 3 is necessary and sufficient. However, we introduce a nonconvex constraint $\mathcal{X}P = I$, which makes our problem more difficult to solve. Furthermore, the inequality (40) is still nonlinear. The next theorem shows how to write these conditions into convex constraints by using Finsler's Lemma again from [14].

THEOREM 1. *There exists a state estimator gain F to solve* (23) *if there exist a symmetric matrix $P \in \mathbf{R}^{2n_x\times 2n_x}$, a scalar $\mu_1 < 0$, and a scalar $\mu_2 < 0$ that satisfy*

$$P > 0 \tag{41}$$

$$\begin{pmatrix} P\mathcal{A}_0 + \mathcal{A}_0^T P & P\mathcal{N} & P\mathcal{D}_0 & P\mathcal{B}_0 \\ \mathcal{N}^T P & -P & 0 & 0 \\ \mathcal{D}_0^T P & 0 & -I & 0 \\ \mathcal{B}_0^T P & 0 & 0 & \mu_1^{-1} I \end{pmatrix} < 0 \tag{42}$$

$$\begin{pmatrix} P\mathcal{A}_0 + \mathcal{A}_0^T P & 0 & P\mathcal{D}_0 & \mathcal{C}_0^T & P\mathcal{N} \\ 0 & -P & 0 & G_0^T & 0 \\ \mathcal{D}_0^T P & 0 & -I & E_0^T & 0 \\ \mathcal{C}_0 & G_0 & E_0 & \mu_2^{-1} I & 0 \\ \mathcal{N}^T P & 0 & 0 & 0 & -P \end{pmatrix} < 0 \tag{43}$$

Proof. The result follows from Lemma 3 and Finsler's lemma. If the inequality (39) holds, it is equivalent to the following: there exists a $\mu_1 \in \mathbf{R}$ such that:

$$\mathcal{A}_0\mathcal{X} + \mathcal{X}\mathcal{A}_0^T + \mathcal{N}\mathcal{X}\mathcal{N}^T + \mathcal{D}_0\mathcal{D}_0^T - \mu_1\mathcal{B}_0\mathcal{B}_0^T < 0$$

Apply the congruence transformation

$$\mathcal{X}^{-1}(\mathcal{A}_0\mathcal{X}+\mathcal{X}\mathcal{A}_0^T+\mathcal{N}\mathcal{X}\mathcal{N}^T+\mathcal{D}_0\mathcal{D}_0^T-\mu_1\mathcal{B}_0\mathcal{B}_0^T)\mathcal{X}^{-1}<0$$

with the definition $P:=\mathcal{X}^{-1}>0$, it yields

$$P\mathcal{A}_0+\mathcal{A}_0^TP+P\mathcal{N}P^{-1}\mathcal{N}^TP+P\mathcal{D}_0\mathcal{D}_0^TP-\mu_1P\mathcal{B}_0\mathcal{B}_0^TP<0$$

With the assumption $\mu_1<0$, the above condition can be equivalently written as (42) by using the Schur complement.

Similarly, if the inequality (40) holds, it is equivalent to the existence of a $\mu_2\in\mathbf{R}$ such that

$$\begin{pmatrix} P\mathcal{A}_0+\mathcal{A}_0^TP+P\mathcal{N}\mathcal{X}\mathcal{N}^TP & 0 & P\mathcal{D}_0 \\ 0 & -P & 0 \\ \mathcal{D}_0^TP & 0 & -I \end{pmatrix} - \mu_2\begin{pmatrix} \mathcal{C}_0^T \\ G_0^T \\ E_0^T \end{pmatrix}(\mathcal{C}_0 \quad G_0 \quad E_0) < 0$$

By assuming $\mu_2<0$ and applying Schur complements twice on the above inequality, it obtains the LMI (43), which completes the proof of the theorem. ■

4 FSN Filter Design

In Sec. 3, a sufficient LMI condition to examine the existence of state estimator for FSN models has been given. This section is dedicated to determine an estimator gain F that additionally guarantees the performance criterion, $\varepsilon_\infty\{\tilde{z}\tilde{z}^T\}<\Omega$. We know that $\varepsilon_\infty\{\tilde{z}\tilde{z}^T\}$ can be computed from

$$\varepsilon_\infty\{\tilde{z}\tilde{z}^T\}=C_z\varepsilon_\infty\{\tilde{x}\tilde{x}^T\}C_z^T=\bar{C}_z\mathcal{X}\bar{C}_z^T \qquad (44)$$

where the state covariance matrix $\mathcal{X}$ is defined in (22) and $\bar{C}_z=C_z[I \quad 0]$. The algorithm to solve the filtering problem can be derived from the following theorem.

THEOREM 2. For a given Ω, if there exist a positive definite symmetric matrix $P\in\mathbf{R}^{2n_x\times 2n_x}$, a scalar $\mu_1<0$, and a scalar $\mu_2<0$ that satisfy (42), (43), and

$$\begin{pmatrix} \Omega & \bar{C}_z \\ \bar{C}_z^T & P \end{pmatrix} > 0 \qquad (45)$$

then there exists a filter gain F such that $\varepsilon_\infty\{\tilde{z}\tilde{z}^T\}<\Omega$. All the solutions F are given by

$$F=-R^{-1}\Gamma^T\Phi\Lambda^T(\Lambda\Phi\Lambda^T)^{-1}+S^{1/2}L(\Lambda\Phi\Lambda^T)^{-1/2} \qquad (46)$$

where

$$S=R^{-1}-R^{-1}\Gamma^T[\Phi-\Phi\Lambda^T(\Lambda\Phi\Lambda^T)^{-1}\Lambda\Phi]\Gamma R^{-1} \qquad (47)$$

L is an arbitrary matrix such that $\|L\|<1$, and R is an arbitrary positive definite matrix that

$$\Phi=(\Gamma R^{-1}\Gamma^T-\Theta)^{-1}>0 \qquad (48)$$

and

$$\Theta=\begin{pmatrix} \mathcal{A}_0P^{-1}+P^{-1}\mathcal{A}_0^T+\mathcal{N}P^{-1}\mathcal{N}^T & 0 & \mathcal{D}_0 \\ 0 & -P^{-1} & 0 \\ \mathcal{D}_0^T & 0 & -I \end{pmatrix}$$

$$\Gamma=\begin{pmatrix} \mathcal{B}_0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Lambda=(\mathcal{C}_0P^{-1} \quad G_0P^{-1} \quad E_0)$$

Proof. The inequality (45) can be manipulated by $\varepsilon_\infty\{\tilde{z}\tilde{z}^T\}=\bar{C}_z\mathcal{X}\bar{C}_z^T<\Omega$, then we can use Schur complement to convert it into a LMI

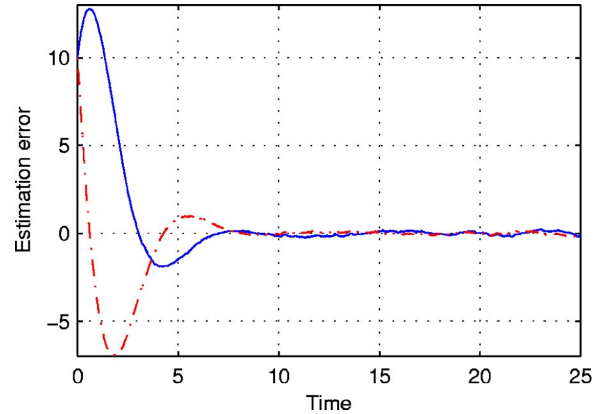

Fig. 1 Estimation error of FSN filter (solid line corresponds to the error of state variable x_1, dashed-dotted line corresponds to the estimation error of state variable x_2)

$$\begin{pmatrix} \Omega & \bar{C}_z \\ \bar{C}_z^T & \mathcal{X}^{-1} \end{pmatrix}=\begin{pmatrix} \Omega & \bar{C}_z \\ \bar{C}_z^T & P \end{pmatrix}>0$$

The proof solving for F follows a similar approach to [14]. ■

We observe that the optimization approach proposed in the above theorem is a convex programming problem stated as LMIs, which can be solved by efficient methods.

4.1 Numerical Example. In order to determine the applicability of the method, an example of filter design is presented next. We will consider a simple mechanical system that consists of a mass, a spring, and a damper. The plant noise and measurement noise are modeled as FSN white noise

$$\begin{cases} \dot{x}=\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}x+\begin{pmatrix} 1 \\ 1 \end{pmatrix}\omega \\ y=(3 \quad 3)x+v \\ z=(1 \quad 1)x \end{cases}$$

For simplicity, we assume that $\Sigma_\omega=\sigma_\omega I$, $\Sigma_v=\sigma_v I$. The noise-to-signal ratio (NSR) is $\sigma_\omega=0.1$, $\sigma_v=0.1$, respectively, and $M=(1 \quad 0.5)$.

The performance criterion for the filter design is $\varepsilon_\infty\{\tilde{z}\tilde{z}^T\}<\Omega$ where $\Omega=4$.

The filter that results from our method is

$$\dot{\hat{x}}=\begin{pmatrix} -0.00895 & 0.99105 \\ -0.99494 & -0.99494 \end{pmatrix}\hat{x}+\begin{pmatrix} 0.002985 \\ -0.001685 \end{pmatrix}y$$

The simulation result shows that the output covariance of the estimation error is

$$\varepsilon_\infty\{\tilde{z}\tilde{z}^T\}=2.9585$$

which satisfies the design requirement, since $2.9585<4$.

The performance of the FSN filter introduced in this paper is illustrated in Fig. 1, where the error of each state variable is plotted. When compared to Fig. 1, Fig. 2 demonstrates the inferiority of the Kalman filter, which ignores the FSN structure of the noise by setting $W=W_0$ and $V=V_0$ in (8) and (9). Note that the peak values of the state error using the standard Kalman filter (from Fig. 2) are approximately 38 and 23, respectively, as compared to peak errors of approximately, 7 and 13, respectively for the FSN estimator.

Another interesting observation is that, in this example, the upper bound Ω is a scalar; therefore, the problem of minimizing the upper bound Ω can also be defined based on Theorem 2. The solution to the problem of minimizing Ω subject to the inequalities (42), (43), and (45) produces the optimal filter that is associ-

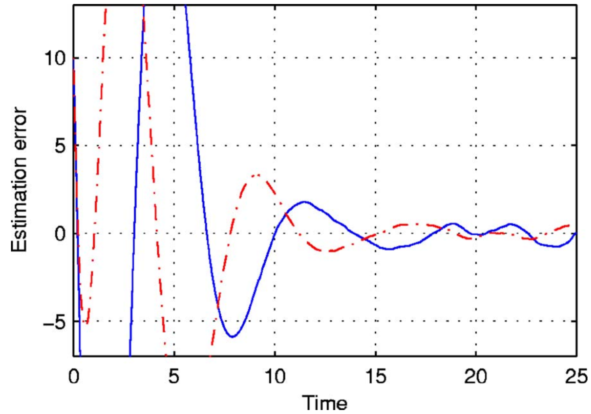

Fig. 2 Estimation error of Kalman filter (solid line corresponds to the error of state variable x_1, dashed-dotted line corresponds to the estimation error of state variable x_2)

ated with the optimal cost. Indeed, Fig. 3 illustrates the tradeoff between the optimal performance and the precision of system devices, where the optimal output variance is plotted as a function of information quality $(\sigma_w+\sigma_v)^{-1}$. Given a certain performance, one can obtain what is the required precision on the sensors and other devices from that plot. As expected, a qualitative interpretation of this plot is to show that better performance requires more information quality.

5 Discrete-Time Systems

In this section, we develop the discrete-time counterpart of the filter design presented in Secs. 3 and 4. Consider the following discrete-time system with state space representation

$$x(k+1) = Ax(k) + D\omega(k)$$

$$y(k) = Cx(k) + v(k)$$

$$z(k) = C_z x(k) \tag{49}$$

where $x(k) \in \mathbf{R}^{n_x}$ is the state variable; $y(k) \in \mathbf{R}^{n_y}$ is the measurement output; $z(k) \in \mathbf{R}^{n_z}$ is the output of interest for performance evaluation; $\omega(k) \in \mathbf{R}^{n_w}$ and $v(k) \in \mathbf{R}^{n_y}$ are zero-mean FSN white noises with unknown covariances W and V respectively; all matrices A, C, C_z, D are given and assumed to have proper dimensions. As in the continuous-time case, we consider that the noise source is modeled according to the FSN assumption. That is, assuming

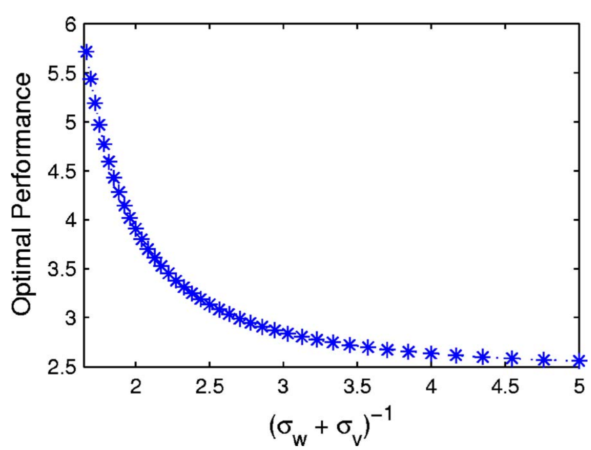

Fig. 3 Optimal performance as a function of information quality

$$\varepsilon_\infty\{\omega(k)\} = 0, \quad \varepsilon_\infty\{\omega(k)\omega(\tau)^T\} = W\delta(k-\tau) \tag{50}$$

$$\varepsilon_\infty\{v(k)\} = 0, \quad \varepsilon_\infty\{v(k)v(\tau)^T\} = V\delta(k-\tau) \tag{51}$$

Suppose the vector $\omega_a(k) \in \mathbf{R}^{n_w}$ describes the signal that is corrupted by the noise $\omega(k)$, and $\omega_a(k)$ is linearly related to state variable $x(k)$

$$\omega_a(k) = Mx(k) \tag{52}$$

where M is given. Assuming that the state covariance matrix $X = \varepsilon_\infty\{x(k)x(k)^T\}$ associated with the system (49) exists, we could compute

$$W = W_0 + \Sigma_\omega M X M^T \Sigma_\omega \tag{53}$$

$$V = V_0 + \Sigma_v C X C^T \Sigma_v \tag{54}$$

where W_0, V_0 are given positive definite constant matrices; $\Sigma_\omega = \mathrm{diag}\{\sigma_{\omega_1}, \sigma_{\omega_2}, \ldots, \sigma_{\omega_{n_\omega}}\}$, $\Sigma_v = \mathrm{diag}\{\sigma_{v_1}, \sigma_{v_2}, \ldots, \sigma_{v_{n_y}}\}$, and $\sigma_{\omega_i}, \sigma_{v_i}$ are noise-to-signal ratio (NSR) of the ith channel, respectively.

Combining the linear filter

$$\hat{x}(k+1) = A\hat{x}(k) + F[y(k) - C\hat{x}(k)] \tag{55}$$

$$\hat{z}(k) = C_z \hat{x}(k) \tag{56}$$

and the estimation error dynamics

$$\tilde{x}(k+1) = (A - FC)\tilde{x}(k) + D\omega(k) - Fv(k) \tag{57}$$

$$\tilde{z}(k) = C_z \tilde{x}(k) \tag{58}$$

where $\hat{x}(k)$ is the unbiased estimate of the state $x(k)$, F is the filter gain to be determined, the estimation error $\tilde{x}(k) = x(k) - \hat{x}(k)$, and $\tilde{z}(k)$ denotes the estimation error of particular interest, we obtain the augmented adjoint system

$$\mathbf{x}(k+1) = \mathcal{A}\mathbf{x}(k) + \mathcal{D}\mathbf{w}(k) \tag{59}$$

where

$$\mathbf{x}(k) = \begin{pmatrix} \tilde{x}(k) \\ \hat{x}(k) \end{pmatrix}, \quad \mathbf{w}(k) = \begin{pmatrix} \omega(k) \\ v(k) \end{pmatrix} \tag{60}$$

$$\mathcal{A} = \begin{pmatrix} A - FC & 0 \\ FC & A \end{pmatrix} = \mathcal{A}_0 + \mathcal{B}_0 F \mathcal{C}_0 \tag{61}$$

$$\mathcal{D} = \begin{pmatrix} D & -F \\ 0 & F \end{pmatrix} = \mathcal{D}_0 + \mathcal{B}_0 F E_0 \tag{62}$$

$$\mathcal{A}_0 = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, \quad \mathcal{B}_0 = \begin{pmatrix} -I \\ I \end{pmatrix} \tag{63}$$

$$\mathcal{C}_0 = (C \quad 0), \quad \mathcal{D}_0 = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}, \quad E_0 = (0 \quad I) \tag{64}$$

The objective in this section is to provide the existence condition of the state estimator for discrete-time FSN systems based on linear matrix inequalities (LMIs). The key idea of this filtering problem is to find the estimate $\hat{x}(k)$ of $x(k)$ such that the performance criterion $\varepsilon_\infty\{\tilde{z}(k)\tilde{z}(k)^T\} < \Omega$ is satisfied for the given Ω.

5.1 Existence Condition. As for continuous-time system, we start by defining the upper bound of the state covariance matrix of system (59) as

$$\mathcal{X} \geq \varepsilon_\infty\{\mathbf{x}(k)\mathbf{x}(k)^T\} \tag{65}$$

if it exists, it should satisfy the following Lyapunov inequality:

$$0 > \mathcal{A}\mathcal{X}\mathcal{A}^T - \mathcal{X} + \mathcal{D}\begin{pmatrix} W & 0 \\ 0 & V \end{pmatrix}\mathcal{D}^T \quad (66)$$

where W and V are symmetric and positive definite. Substitution of (53) and (54) and (61) and (62) into the above inequality yields

$$\begin{aligned} 0 > & (\mathcal{A}_0 + \mathcal{B}_0 F \mathcal{C}_0)\mathcal{X}(\mathcal{A}_0 + \mathcal{B}_0 F \mathcal{C}_0)^T - \mathcal{X} + \mathcal{N}\mathcal{X}\mathcal{N}^T \\ & + (\mathcal{B}_0 F G_0)\mathcal{X}(\mathcal{B}_0 F G_0)^T + (\mathcal{D}_0 + \mathcal{B}_0 F E_0)\mathcal{W}(\mathcal{D}_0 + \mathcal{B}_0 F E_0)^T \end{aligned} \quad (67)$$

where

$$\mathcal{N} = \begin{pmatrix} D\Sigma_\omega M & D\Sigma_\omega M \\ 0 & 0 \end{pmatrix} \quad (68)$$

$$\mathcal{W} = \begin{pmatrix} W_0 & 0 \\ 0 & V_0 \end{pmatrix} \quad (69)$$

$$G_0 = (-\Sigma_v C \quad -\Sigma_v C) \quad (70)$$

LEMMA 4. The inequality (67) can be rewritten in a form

$$\Gamma F \Lambda + (\Gamma F \Lambda)^T + \Theta < 0 \quad (71)$$

where

$$\Theta = \begin{pmatrix} -\mathcal{X} + \mathcal{N}\mathcal{X}\mathcal{N}^T & 0 & \mathcal{A}_0\mathcal{X} & \mathcal{D}_0\mathcal{W} \\ 0 & -\mathcal{X} & 0 & 0 \\ \mathcal{X}\mathcal{A}_0^T & 0 & -\mathcal{X} & 0 \\ \mathcal{W}\mathcal{D}_0^T & 0 & 0 & -\mathcal{W} \end{pmatrix} \quad (72)$$

$$\Gamma = \begin{pmatrix} \mathcal{B}_0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (73)$$

$$\Lambda = (0 \quad G_0\mathcal{X} \quad \mathcal{C}_0\mathcal{X} \quad E_0\mathcal{W}) \quad (74)$$

The proof can be obtained following the same steps as used in the proof of Lemma 1. It is important to note that the filtering design problem has been converted into a search for the solution F in inequality (71). Using the same techniques as in Sec. 3, one can obtain the existence conditions of the state estimator and a parametrization of all admissible solutions.

LEMMA 5. Condition (71) is equivalent to the following statement: there exist symmetric positive definite matrices $\mathcal{X}, P \in \mathbf{R}^{2n_x \times 2n_x}$ that satisfy

$$\mathcal{X}P = I \quad (75)$$

$$\mathcal{B}_0^{\perp}(\mathcal{A}_0\mathcal{X}\mathcal{A}_0^T - \mathcal{X} + \mathcal{N}\mathcal{X}\mathcal{N}^T + \mathcal{D}_0\mathcal{W}\mathcal{D}_0^T)\mathcal{B}_0^{\perp T} < 0 \quad (76)$$

$$\begin{pmatrix} 0 \\ G_0^T \\ \mathcal{C}_0^T \\ E_0^T \end{pmatrix}^{\perp} \begin{pmatrix} -P + P\mathcal{N}\mathcal{X}\mathcal{N}^T P & 0 & P\mathcal{A}_0 & P\mathcal{D}_0 \\ 0 & -P & 0 & 0 \\ \mathcal{A}_0^T P & 0 & -P & 0 \\ \mathcal{D}_0^T P & 0 & 0 & -\mathcal{W}^{-1} \end{pmatrix} \begin{pmatrix} 0 \\ G_0^T \\ \mathcal{C}_0^T \\ E_0^T \end{pmatrix}^{\perp T} < 0 \quad (77)$$

This lemma also relies on the use of Schur complement and on Finsler's lemma. Its proof can be easily obtained following the same steps as used in Lemma 3. Lemma 5 provides the necessary and sufficient condition for the existence of the state estimator. However, the problem becomes more difficult to solve after introducing a nonconvex constraint $\mathcal{X}P = I$, and the inequality (77) needs to be constructed as a LMI. Using techniques that parallel the developments performed in Sec. 3, it is possible to rewrite these conditions into convex constraints by applying Finsler's lemma for discrete-time FSN systems. Again the proof is omitted for brevity.

THEOREM 3. A state estimator gain F solving (67) exists if there is a symmetric matrix $P \in \mathbf{R}^{2n_x \times 2n_x}$, a scalar $\mu_1 < 0$, and a scalar $\mu_2 < 0$ that satisfy

$$P > 0 \quad (78)$$

$$\begin{pmatrix} -P & P\mathcal{A}_0 & P\mathcal{N} & P\mathcal{D}_0 & P\mathcal{B}_0 \\ \mathcal{A}_0^T P & -P & 0 & 0 & 0 \\ \mathcal{N}^T P & 0 & -P & 0 & 0 \\ \mathcal{D}_0^T P & 0 & 0 & -\mathcal{W}^{-1} & 0 \\ \mathcal{B}_0^T P & 0 & 0 & 0 & \mu_1^{-1} I \end{pmatrix} < 0 \quad (79)$$

$$\begin{pmatrix} -P & 0 & P\mathcal{A}_0 & P\mathcal{D}_0 & 0 & P\mathcal{N} \\ 0 & -P & 0 & 0 & G_0^T & 0 \\ \mathcal{A}_0^T P & 0 & -P & 0 & \mathcal{C}_0^T & 0 \\ \mathcal{D}_0^T P & 0 & 0 & -\mathcal{W}^{-1} & E_0^T & 0 \\ 0 & G_0 & \mathcal{C}_0 & E_0 & \mu_2^{-1} I & 0 \\ \mathcal{N}^T P & 0 & 0 & 0 & 0 & -P \end{pmatrix} < 0 \quad (80)$$

5.2 Filter Design. In Sec. 5.1, a sufficient LMI condition for checking the existence of state estimator has been given. Here, we provide conditions that guarantee the additional closed loop system performance. We will determine a state estimator F such that the performance criterion, $\varepsilon_\infty\{\tilde{z}(k)\tilde{z}(k)^T\} < \Omega$, is satisfied. The fundamental algorithm that enables us to solve the filtering problem is derived from Theorem 4.

THEOREM 4. For a given Ω, if there exist a positive definite symmetric matrix $P \in \mathbf{R}^{2n_x \times 2n_x}$, a scalar $\mu_1 < 0$, and a scalar $\mu_2 < 0$ that satisfy (79), (80), and

$$\begin{pmatrix} \Omega & \bar{C}_z \\ \bar{C}_z^T & P \end{pmatrix} > 0 \quad (81)$$

where

$$\bar{C}_z = C_z[I \quad 0] \quad (82)$$

then there exists a state estimator gain F such that $\varepsilon_\infty\{\tilde{z}(k)\tilde{z}(k)^T\} < \Omega$. All the solutions F are given by

$$F = -R^{-1}\Gamma^T\Phi\Lambda^T(\Lambda\Phi\Lambda^T)^{-1} + S^{1/2}L(\Lambda\Phi\Lambda^T)^{-1/2} \quad (83)$$

where

$$S = R^{-1} - R^{-1}\Gamma^T[\Phi - \Phi\Lambda^T(\Lambda\Phi\Lambda^T)^{-1}\Lambda\Phi]\Gamma R^{-1} \quad (84)$$

L is an arbitrary matrix such that $\|L\| < 1$ and R is an arbitrary positive definite matrix such that

$$\Phi = (\Gamma R^{-1}\Gamma^T - \Theta)^{-1} > 0 \quad (85)$$

and

$$\Theta = \begin{pmatrix} -P^{-1} + \mathcal{N}P^{-1}\mathcal{N}^T & 0 & \mathcal{A}_0 P^{-1} & \mathcal{D}_0\mathcal{W} \\ 0 & -P^{-1} & 0 & 0 \\ P^{-1}\mathcal{A}_0^T & 0 & -P^{-1} & 0 \\ \mathcal{W}\mathcal{D}_0^T & 0 & 0 & -\mathcal{W} \end{pmatrix}$$

$$\Gamma = \begin{pmatrix} \mathcal{B}_0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \Lambda = (0 \quad G_0 P^{-1} \quad \mathcal{C}_0 P^{-1} \quad E_0\mathcal{W})$$

This result establishes the counterpart of Theorem 2 for discrete-time FSN systems. As in the continuous-time case, the

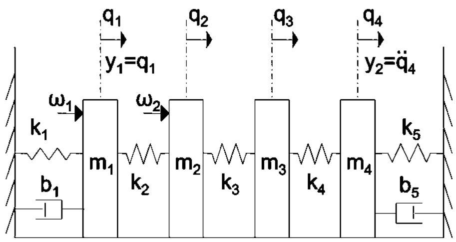

Fig. 4 Four mass mechanical system with springs and dampers

optimization approach proposed in this theorem is a convex programming problem expressed in forms of LMIs, which can be solved by many efficient methods.

5.3 Numerical Examples. In order to determine the applicability of the method, two examples to solve for the system design are presented next.

5.3.1 Four Mass Mechanical System. Consider the four mass mechanical system with springs and dampers depicted in Fig. 4. The discrete-time system dynamics is described in the following state space form

$$x(k+1)=\begin{bmatrix} I & \Delta I \\ -\Delta M^{-1}K & I-\Delta M^{-1}G \end{bmatrix}x(k)+\begin{bmatrix} 0 \\ \Delta M^{-1}D \end{bmatrix}\omega(k) \tag{86}$$

with the measurement

$$y(k)=Cx(k)+v(k) \tag{87}$$

and the desired output

$$z(k)=C_z x(k) \tag{88}$$

where Δ is the time step (0.01 s) and

$$x(k)=\begin{pmatrix} q \\ \dot{q} \end{pmatrix}, \quad y(k)=\begin{pmatrix} q_1 \\ \ddot{q}_4 \end{pmatrix}, \quad \omega(k)=\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$$

$$q=[q_1 \quad q_2 \quad q_3 \quad q_4]^T$$

$$M=\begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & m_3 & 0 \\ 0 & 0 & 0 & m_4 \end{bmatrix}, \quad G=\begin{bmatrix} b_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_5 \end{bmatrix}$$

$$K=\begin{bmatrix} k_1+k_2 & -k_2 & 0 & 0 \\ -k_2 & k_2+k_3 & -k_3 & 0 \\ 0 & -k_3 & k_3+k_4 & -k_4 \\ 0 & 0 & -k_4 & k_4+k_5 \end{bmatrix}$$

$$D=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}^T$$

$$C_z=C=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_4/m_4 & -(k_4+k_5)/m_4 & 0 & 0 & 0 & -b_5/m_4 \end{bmatrix}$$

$$m_1=m_2=m_4=1, \quad m_3=2, \quad b_1=5, \quad b_5=2$$

$$k_1=k_3=k_4=1, \quad k_2=2, \quad k_5=4$$

Note that $\omega(k)$ and $v(k)$ are modeled as FSN white noises with covariance

$$W=\begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix}, \quad W_1=\sigma_w^2\varepsilon_\infty\{x_5(k)x_5(k)^T\},$$

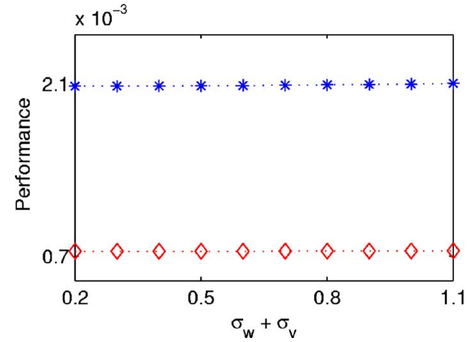

Fig. 5 Performance as a function of $\sigma_w+\sigma_v$ corresponding to the two different noise contributions (Star curve corresponds to the output covariance of the first estimation error, diamond curve corresponds to the output covariance of the second estimation error)

$$W_2=\sigma_w^2\varepsilon_\infty\{x_6(k)x_6(k)^T\}, \quad \sigma_w=0.1$$

$$V=\sigma_v^2 C\varepsilon_\infty\{x(k)x(k)^T\}C^T, \quad \sigma_v=0.1$$

The performance criterion for the filter design is $[\varepsilon_\infty\{\tilde{z}(k)\tilde{z}(k)^T\}]_{i,i}<0.1$ $(i=1,2)$. Figure 5 demonstrates the performance of the FSN state estimator introduced in this paper. We know that the signal-to-noise ratio of all measurement and control devices plays an important role in certain properties of a FSN model. With the same specified performance requirement, we design a state estimator for each different pair of σ_w and σ_v. The simulation results show that the output covariances of the estimation errors only change within a very small amount of value, the state estimators we design can always satisfy the performance criterion.

Remark 1. The comparison between the FSN filter and the Kalman filter for the discrete-time system demonstrates the same result as obtained in continuous case, hence the simulation results for the discrete-time system is omitted for brevity. Since the FSN system allows the noise variance to be affinely related to the variance of the signal corrupted by the noise, such system has properties that system with traditional noise sources cannot possess, i.e., improving performance in the presence of a finite-precision computing environment [4].

5.3.2 Biomechanical Hand Movement System. Consider the hand modeled as a point mass ($m=1$ kg) whose one-dimensional position at time t is $p(t)$, and the velocity at time t is $v(t)$. The combined action of all muscles is represented with the force $f(t)$ acting on the hand. The control signal $u(t)$ is transformed into force by adding control-dependent noise and applying a second-order musclelike low-pass filter

$$\tau_1\tau_2\ddot{f}(t)+(\tau_1+\tau_2)\dot{f}(t)+f(t)=u(t)$$

where $\tau_1=\tau_2=0.04$ s. We know that the above filter can be written as a pair of coupled first-order filters

$$\tau_1\dot{g}+g=u, \quad \tau_2\dot{f}+f=g$$

The sensory feedback carries the information about position, velocity, and force. The discrete-time system dynamics is described as follows:

$$x(k+1)=Ax(k)+B[1+\sigma_c\varepsilon(k)]u(k)+\omega(k)$$

$$y(k)=Cx(k)+v(k)$$

$$z(k)=C_z x(k) \tag{89}$$

where

$$x(k)=(p(k) \quad v(k) \quad f(k) \quad g(k))^T, \quad y(k)=(p(k) \quad v(k) \quad f(k))^T$$

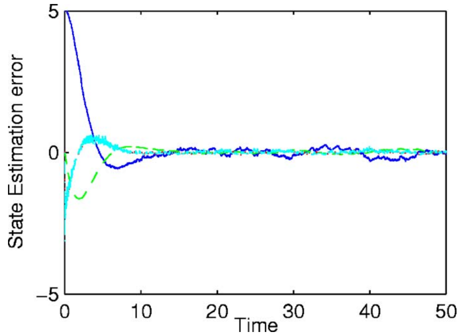

Fig. 6 Estimation error for hand movement system, each curve corresponds to the error of each state variable

$$A=\begin{bmatrix}1 & \Delta & 0 & 0\\ 0 & 1 & \Delta/m & 0\\ 0 & 0 & 1-\Delta/\tau_2 & \Delta/\tau_2\\ 0 & 0 & 0 & 1-\Delta/\tau_1\end{bmatrix},\quad B=\begin{bmatrix}0\\0\\0\\ \Delta/\tau_1\end{bmatrix}$$

$$C=C_z=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}$$

and $\omega(k), v(k), \varepsilon(k)$ are independent zero-mean Gaussian white noise sequences with covariance

$$\Omega^{\omega}=(\mathrm{diag}[0.01,0.001,0.01,0.01])^2,$$

$$\Omega^{v}=(\mathrm{diag}[0.01,0.1,0.5])^2,\quad \Omega^{\varepsilon}=I$$

Note that $\sigma_c=0.5$ is a unitless quantity that defines the noise magnitude relative to the control signal magnitude. And the time step $\Delta=0.01$ s.

Given a controller

$$u(k)=\begin{bmatrix}-1.6032 & -3.0297 & -0.3361 & -2.7793\end{bmatrix}x(k)$$

such that the system (89) is mean square stable, the objective is to find a state estimator that bounds the estimation error below a specified error covariance: $[\varepsilon_\infty\{\tilde{z}(k)\tilde{z}(k)^T\}]_{i,i}<\Omega \quad (i=1,2,3)$, where $\Omega=0.1$.

Figure 6 illustrates the performance of the filter introduced in this paper, where the error of each state variable is plotted. The simulation result shows that the output covariance of the estimation error are $[\varepsilon_\infty\{\tilde{z}(k)\tilde{z}(k)^T\}]_{1,1}=0.0198$, $[\varepsilon_\infty\{\tilde{z}(k)\tilde{z}(k)^T\}]_{2,2}=0.0037$, $[\varepsilon_\infty\{\tilde{z}(k)\tilde{z}(k)^T\}]_{3,3}=0.0018<\mathbf{0.1}$, which satisfy the design requirement.

Remark 2. Here we add a mild additional constraint to make the estimation design for FSN systems into a convex problem. Although the scaler $\mu_1<0$ and $\mu_2<0$ are in a sense conservative for the design purpose, from the many numerical illustrations in the paper, it demonstrates that μ_1, μ_2 negative is not restrictive in the filtering problem.

6 Conclusions

FSN noise models are more practical than normal white noise models, since they allow the size (intensity) of the noises to be affinely related to the size (variance) of the signals they corrupt. Such noises are found in digital signal processing with both fixed- and floating-point arithmetic. Such models are found in analog sensors and actuators that produce more noise when the power supplies in these devices must provide more power (for an increased dynamic range of the signals in the estimation or control problem).

This paper derives sufficient conditions for the existence of the state estimator with FSN noise models. By adding a mild constraint, the original problem (of estimating to within a specified covariance error bound), is solved as a convex problem. Associated with the solvable convex conditions, a LMI-based approach is examined for the design of the estimator with the FSN model. This estimator design guarantees the performance requirement and the design algorithm is convergent [15–26].

Acknowledgment

The authors gratefully acknowledge the support of NIH Grant No. R01-NS045915.

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